

Proof Theory of Constructive Systems: Inductive Types and Univalence

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Abstract

In Feferman’s work, explicit mathematics and theories of generalized inductive definitions play a central role. One objective of this article is to describe the connections with Martin-Löf type theory and constructive Zermelo-Fraenkel set theory. Proof theory has contributed to a deeper grasp of the relationship between different frameworks for constructive mathematics. Some of the reductions are known only through ordinal-theoretic characterizations. The paper also addresses the strength of Voevodsky’s univalence axiom.

A further goal is to investigate the strength of intuitionistic theories of generalized inductive definitions in the framework of intuitionistic explicit mathematics that lie beyond the reach of Martin-Löf type theory.

Key words: *Explicit mathematics, constructive Zermelo-Fraenkel set theory, Martin-Löf type theory, univalence axiom*

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1 Introduction

Intuitionistic systems of inductive definitions have figured prominently in Solomon Feferman’s program of reducing classical subsystems of analysis and theories of iterated inductive definitions to constructive theories of various kinds. In the special case of classical theories of finitely as well as transfinitely iterated inductive definitions, where the iteration occurs along a computable well-ordering, the program was mainly completed by Buchholz, Pohlers, and Sieg more than 30 years ago (see [14, 20]). For stronger theories of inductive definitions such as those based on Feferman’s intuitionistic *Explicit Mathematics*¹ (\mathbf{T}_0^i) some answers have been provided in the last 10 years while some questions are still open.

The aim of the first part of this paper is to survey the landscape of some prominent constructive theories that emerged in the 1970s. In addition to Feferman’s \mathbf{T}_0^i , Myhill’s *Constructive Set Theory* (**CST**) and Martin-Löf type theory (**MLTT**) have been proposed with the aim of isolating the principles on which constructive mathematics is founded, notably the notions of constructive function and set in Bishop’s mathematics.

Martin-Löf type theory with infinitely many universes and inductive types (**W-types**) has attracted a great deal of attention recently because of a newly found connection between type theory and topology, called *homotopy type theory* (**HoTT**), where types are interpreted as spaces, terms as maps and the inhabitants of the iterated identity types on a given type A are viewed as paths, homotopies and higher homotopies of increasing levels, respectively, endowing each type with a weak ω -groupoid structure.

Homotopy type theory, so it appears, has now reached the mathematical mainstream:

¹Feferman introduced the theory of explicit mathematics in [21]. There it was based on intuitionistic logic and notated by \mathbf{T}_0 . The same notation is used e.g. in [14, 37, 59] but increasingly \mathbf{T}_0 came to be identified with its classical version. As a result, we adopt the notation \mathbf{T}_0^i to stress its intuitionistic basis and reserve \mathbf{T}_0 for the classical theory.

Voevodsky's Univalent Foundations require not just one inaccessible cardinal but an infinite string of cardinals, each inaccessible from its predecessor. (M. Harris, *Mathematics without apologies*, 2015).

By Univalent Foundations Harris seems to refer to **MLTT** plus Voevodsky's *Univalence Axiom* (**UA**). That an infinite sequence of inaccessible cardinals is required is a pretty strong statement. Recent research by Bezem, Huber, and Coquand (see [11]), though, indicates that **MLTT**+**UA** has an interpretation in **MLTT** and therefore is proof-theoretically not stronger than **MLTT**. But what is the strength of **MLTT**? As there doesn't seem to exist much common knowledge among type theorists about the strength of various systems and how they relate to the other constructive frameworks as well as classical theories used as a classification hierarchy in reverse mathematics and set theory, it seems reasonable to devote a section to mapping out the relationships and gathering current knowledge in one place. In this section attention will also be paid to the methods employed in proofs such as interpretations but with a particular eye toward the role of ordinal analysis therein.

The second part of this paper (Section 8) will be concerned with extensions of explicit mathematics by principles that allow the construction of inductive classifications that lie way beyond **MLTT**'s reach but still have a constructive flavor. The basic theory here is intuitionistic explicit mathematics \mathbf{T}_0^i . In \mathbf{T}_0^i one can freely talk about monotone operations on classifications and assert the existence of least fixed points of such operators. There are two ways in which one can add a principle to \mathbf{T}_0^i postulating the existence of least fixed points. **MID** merely existentially asserts that every monotone operation has a least fixed point whereas **UMID** not only postulates the existence of a least solution, but, by adjoining a new functional constant to the language, ensures that a fixed point is uniformly presentable as a function of the monotone operation.

The question of the strength of systems of explicit mathematics with **MID** and **UMID** was raised by Feferman in [23]; we quote:

*What is the strength of $\mathbf{T}_0 + \mathbf{MID}$? [...] I have tried, but did not succeed, to extend my interpretation of \mathbf{T}_0 in $\Sigma_2^1 - AC + BI$ to include the statement **MID**. The theory $\mathbf{T}_0 + \mathbf{MID}$ includes all constructive formulations of iteration of monotone inductive definitions of which I am aware, while \mathbf{T}_0 (in its *IG* axiom) is based squarely on the general iteration of accessibility inductive definitions. Thus it would be of great interest for the present subject to settle the relationship between these theories.* (p. 88)

As it turned out, the principles **MID** and even more **UMID** encapsulate considerable strength, when considered on the basis of classical \mathbf{T}_0 . For instance $\mathbf{T}_0 + \mathbf{UMID}$ embodies the strength of Π_2^1 -comprehension. The first (significant) models of $\mathbf{T}_0 + \mathbf{MID}$ were found by Takahashi [87]. Research on the precise strength was conducted by Rathjen [72, 73, 74] and Glaß, Rathjen, Schlüter [27]. The article [75] provides a survey of the classical case. Tupailo [93] obtained the first result in the intuitionistic setting. This and further results will be the topic of section 3.

2 Some Background on Feferman's \mathbf{T}_0^i

The theory of explicit mathematics, here denoted by \mathbf{T}_0^i , is a formal framework that has great expressive power. It is suitable for representing Bishop-style constructive mathematics as well as generalized recursion, including direct expression of structural concepts which admit self-application. Feferman was led to the development of his *explicit mathematics* when trying to understand what Errett Bishop had achieved in his groundbreaking constructive redevelopment of analysis in [12]. For a detailed account see [21, 22]. The ontology behind the axioms of \mathbf{T}_0^i is that the universe of mathematical objects is populated by (a) natural numbers, (b) operations (in general partial) and (c) classifications (akin to Bishop's sets) where operations and classifications are to be understood as given intensionally. Operations can be applied to any object including operations and classifications; they are governed by axioms giving them the structure of a partial combinatory algebra. There are, for example, operations that act on classifications X, Y to produce their

Cartesian product $X \times Y$ and exponential X^Y . The formation of classifications is governed by the *Join*, *Inductive Generation* and *Elementary Comprehension Axiom*.

The language of \mathbf{T}_0^i , $\mathfrak{L}(\mathbf{T}_0^i)$, has two sorts of variables. The free and bound variables (a, b, c, \dots and x, y, z, \dots) are conceived to range over the whole constructive universe which comprises *operations* and *classifications* among other kinds of entities; while upper-case versions of these A, B, C, \dots and X, Y, Z, \dots are used to represent free and bound classification variables.

\mathbf{N} is a classification constant taken to define the class of natural numbers. $\mathbf{0}$, $\mathbf{s}_\mathbf{N}$ and $\mathbf{p}_\mathbf{N}$ are operation constants whose intended interpretations are the natural number 0 and the successor and predecessor operations. Additional operation constants are \mathbf{k} , \mathbf{s} , \mathbf{d} , \mathbf{p} , \mathbf{p}_1 and \mathbf{p}_1 for the two basic combinators, definition by cases on \mathbf{N} , pairing and the corresponding two projections. Additional classification constants are generated using the axioms and the constants \mathbf{j} , \mathbf{i} and $\mathbf{c}_\mathbf{n}$ ($\mathbf{n} < \omega$) for *join*, *induction* and *comprehension*.

There is no arity associated with the various constants. The *terms* of \mathbf{T}_0^i are just the variables and constants of the two sorts. The atomic formulae of \mathbf{T}_0^i are built up using the terms and three primitive relation symbols $=$, \mathbf{App} and ε as follows. If q, r, r_1, r_2 are terms, then $q = r$, $\mathbf{App}(q, r_1, r_2)$, and $q \varepsilon r$ (where r has to be a classification variable or constant) are atomic formulae. $\mathbf{App}(q, r_1, r_2)$ expresses that the operation q applied to r_1 yields the value r_2 ; $q \varepsilon r$ asserts² that q is in r or that q is classified under r .

We write $t_1 t_2 \simeq t_3$ for $\mathbf{App}(t_1, t_2, t_3)$.

The set of formulae are then obtained from these using the propositional connectives and the two quantifiers of each sort.

In order to facilitate the formulation of the axioms, the language of \mathbf{T}_0^i is expanded definitionally with the symbol \simeq and the auxiliary notion of an *application term* is introduced. The set of application terms is given by two clauses:

1. all terms of \mathbf{T}_0^i are application terms; and
2. if s and t are application terms, then (st) is an application term.

If s is an application term and u is a bound or free variable we define $s \simeq u$ by induction on the buildup of s :

$$s \simeq u \quad \text{is} \quad \begin{cases} s = u, & \text{if } s \text{ is a variable or a constant,} \\ \exists x, y [s_1 \simeq x \wedge s_2 \simeq y \wedge \mathbf{App}(x, y, u)] & \text{if } s \text{ is an application term } (s_1 s_2) \end{cases}$$

For s and t application terms, we have auxiliary, defined formulae of the form:

$$s \simeq t \quad := \quad \forall y (s \simeq y \leftrightarrow t \simeq y).$$

Some abbreviations are $t_1 \dots t_n$ for $((\dots(t_1 t_2) \dots) t_n)$; $t \downarrow$ for $\exists y (t \simeq y)$ and $\phi(t)$ for $\exists y (t \simeq y \wedge \phi(y))$.

Gödel numbers for formulae play a key role in the axioms introducing the classification constants $\mathbf{c}_\mathbf{n}$. A formula is said to be *elementary* if it contains only free occurrences of classification variables A (i.e., only as *parameters*), and even those free occurrences of A are restricted: A must occur only to the right of ε in atomic formulas. The Gödel number $\mathbf{c}_\mathbf{n}$ above is the Gödel number of an elementary formula. We assume that a standard Gödel numbering has been chosen for $\mathfrak{L}(\mathbf{T}_0^i)$; if ϕ is an elementary formula and $a, b_1, \dots, b_m, A_1, \dots, A_n$ is a list of variables which includes all parameters of ϕ , then $\{x : \phi(x, b_1, \dots, b_m, A_1, \dots, A_n)\}$ stands for $\mathbf{c}_\mathbf{n}(b_1, \dots, b_m, A_1, \dots, A_n)$; \mathbf{n} is the code of the pair of Gödel numbers $\langle \ulcorner \phi \urcorner, \ulcorner (a, b_1, \dots, b_m, A_1, \dots, A_n) \urcorner \rangle$ and is called the ‘index’ of ϕ and the list of variables.

Some further conventions are useful. Systematic notation for *n-tuples* is introduced as follows: (t) is t , (s, t) is \mathbf{pst} , and (t_1, \dots, t_n) is defined by $((t_1, \dots, t_{n-1}), t_n)$. Finally, t' is written for the term $\mathbf{s}_\mathbf{N} t$, and \perp is the elementary formula $\mathbf{0} \simeq \mathbf{0}'$.

\mathbf{T}_0^i ’s logic is intuitionistic two-sorted predicate logic with identity. Its non-logical axioms are:

²It should be pointed out that we use the symbol “ ε ” instead of “ \in ” deliberately, the latter being reserved for the set-theoretic elementhood relation.

I. Basic Axioms

1. $\forall X \exists x (X = x)$
2. $\text{App}(a, b, c_1) \wedge \text{App}(a, b, c_2) \rightarrow c_1 = c_2$

II. App Axioms

1. $(\mathbf{k}ab) \downarrow \wedge \mathbf{k}ab \simeq a,$
2. $(\mathbf{s}ab) \downarrow \wedge \mathbf{s}abc \simeq ac(bc),$
3. $(\mathbf{p}a_1a_2) \downarrow \wedge (\mathbf{p}_1a) \wedge (\mathbf{p}_2a) \downarrow \wedge \mathbf{p}_i(\mathbf{p}a_1a_2) \simeq a_i$ for $i = 0, 1,$
4. $(c_1 = c_2 \vee c_1 \neq c_2) \wedge (\mathbf{d}abc_1c_2) \downarrow \wedge (c_1 = c_2 \rightarrow \mathbf{d}abc_1c_2 \simeq a) \wedge (c_1 \neq c_2 \rightarrow \mathbf{d}abc_1c_2 \simeq b),$
5. $a \varepsilon \mathbb{N} \wedge b \varepsilon \mathbb{N} \rightarrow [a' \downarrow \wedge \mathbf{p}_0(a') \simeq a \wedge \neg(a' \simeq 0) \wedge (a' \simeq b' \rightarrow a \simeq b)].$

III. Classification Axioms

Elementary Comprehension Axiom (ECA)

$$\exists X [X \simeq \{x : \psi(x)\} \wedge \forall x (x \varepsilon X \leftrightarrow \psi(x))]$$

for each elementary formula ψa , which may contain additional parameters.

Natural Numbers

- (i) $0 \varepsilon \mathbf{N} \wedge \forall x (x \varepsilon \mathbf{N} \rightarrow x' \varepsilon \mathbf{N})$
- (ii) $\phi(0) \wedge \forall x (\phi(x) \rightarrow \phi(x')) \rightarrow (\forall x \varepsilon \mathbf{N}) \phi(x)$ for each formula ϕ of $\mathfrak{L}(\mathbf{T}_0^i)$.

Join (J)

$$\forall x \varepsilon A \exists Y f x \simeq Y \rightarrow \exists X [X \simeq \mathbf{j}(A, f) \wedge \forall z (z \varepsilon X \leftrightarrow \exists x \varepsilon A \exists y (z \simeq (x, y) \wedge y \varepsilon f x))]$$

Inductive Generation (IG)

$$\begin{aligned} \exists X [X \simeq \mathbf{i}(A, B) \wedge \forall x \varepsilon A [\forall y [(y, x) \varepsilon B \rightarrow y \varepsilon X] \rightarrow x \varepsilon X] \\ \wedge [\forall x \varepsilon A [\forall y ((y, x) \varepsilon B \rightarrow \phi(y)) \rightarrow \phi(x)] \rightarrow \forall x \varepsilon X \phi(x)]] \end{aligned}$$

where ϕ is an arbitrary formula of \mathbf{T}_0^i .

3 Type theories

The type theory of Martin-Löf from the 1984 book [49] will be notated by $\mathbf{MLTT}^{\text{ext}}$ where the superscript is meant to convey that this is an *extensional* theory. It has all the usual type constructors $\Pi, \Sigma, +, \mathbf{0}, \mathbf{1}, \mathbf{2}, \text{Id}, \mathbf{W}$ for dependent products, dependent sums, disjoint unions, empty type, unit type, Booleans, propositional identity types, and \mathbf{W} -types, respectively. Moreover, the system comprises a sequence of universe types $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$ externally indexed by the natural numbers. The universe types are closed under the type constructors from the first list and they form a cumulative hierarchy in that \mathcal{U}_n is a type in \mathcal{U}_{n+1} and if A is a type in \mathcal{U}_n then A is also a type in \mathcal{U}_{n+1} .

In the version of [49] the identity type was taken to be extensional whereas in the more recent versions, e.g. [54] and the one forming the basis for homotopy type theory (see [36]), it is considered to be intensional. The intensional version will simply be denoted by \mathbf{MLTT} . For the proof-theoretic strength, though, it turns out that the difference is immaterial. The reasons will be explained below, but perhaps a first good approximation comes from the observation that (exact) lower bounds can be established by interpreting

certain set theories in type theory in such a way that the extensional identity type can be dispensed with in these interpretations, although for validating certain forms of the axiom of choice, e.g. the **$\Pi\Sigma W\text{-AC}$** axiom to be discussed below, chunks of extensionality are still required. Since we shall be discussing (partial) conservativity results of extensional over intensional type theory below, let's recall the differences.

Definition 3.1 A key feature of Martin-Löf's type theory is the distinction of two notions of identity (or equality). *Judgemental identity* appears in judgements in the two forms $\Gamma \vdash s = t : A$ and $\Gamma \vdash A = B$ type between terms and between types, respectively. The general equality rules (reflexivity, symmetry, transitivity) and substitution rules, simultaneously at the level of terms and types, apply to these judgements as further inference rules.³ But there is also *propositional identity* which gives rise to types $\text{Id}(A, s, t)$ and allows for internal reasoning about identity.

The rules for the extensional identity type are the following:⁴

$$\begin{array}{ll}
(\text{Id-Formation}) & \frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A}{\Gamma \vdash \text{Id}(A, a, b) \text{ type}} \\
(\text{Id-Introduction}) & \frac{\Gamma \vdash a : A}{\Gamma \vdash \text{refl}(a) : \text{Id}(A, a, a)} \\
(\text{Id-Uniqueness}) & \frac{\Gamma \vdash p : \text{Id}(A, a, b)}{\Gamma \vdash p = \text{refl}(a) : \text{Id}(A, a, b)} \\
(\text{Id-Reflection}) & \frac{\Gamma \vdash p : \text{Id}(A, a, b)}{\Gamma \vdash a = b : A}.
\end{array}$$

Reflection has the effect of rendering judgemental identity undecidable, i.e., the (type checking) questions whether $\Gamma \vdash a = b : A$ or $\Gamma \vdash a : A$ hold become undecidable. On the other hand, the set-theoretic models and many recursion-theoretic models of type theory (see [6, 9, 59]) validate extensionality, lending it an intuitive appeal.

For the intensional identity type, the foregoing rules of formation and introduction are retained, however, uniqueness and reflection are jettisoned, getting replaced by elimination and equality rules which are motivated by Leibniz's principle of indiscernibility, namely that identical elements are those that satisfy the same properties. Though instead of capturing identity by quantifying (impredicatively) over all properties (as in Principia), the entire family of identity types $(\text{Id}(A, x, y))_{x, y : A}$ is viewed as being inductively generated with sole constructor refl (see [54, 36]). The elimination and equality rules are the following:

$$\begin{array}{ll}
(\text{Id-Elimination}) & \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma \vdash b : A \\ \Gamma \vdash c : \text{Id}(A, a, b) \\ \Gamma, x : A, y : A, z : \text{Id}(A, x, y) \vdash C(x, y, z) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, \text{refl}(x)) \end{array}}{\Gamma \vdash J(c, d) : C(a, b, c)} \\
(\text{Id-Equality}) & \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma, x : A, y : A, z : \text{Id}(A, x, y) \vdash C(x, y, z) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, \text{refl}(x)) \end{array}}{\Gamma \vdash J(\text{refl}(a), d) = d(a) : C(a, a, \text{refl}(a))}.
\end{array}$$

³See [54, Ch.5] or [36, A.2.2], where they are called structural rules.

⁴The rules are essentially the ones used in [49], except that [49] has a constant r as the sole canonical element of all inhabited types $\text{Id}(A, a, b)$. Here we use $\text{refl}(a)$ to make the comparison with the intensional case more transparent. In [49], Id-Uniqueness and Id-Reflection are called I-equality and I-elimination, respectively.

An immediate consequence of these rules is the indiscernibility of identical elements expressed as follows. For every family $(C(x))_{x:A}$ of types there is a function

$$f : \prod_{x,y:A} \prod_{p:\text{Id}(A,x,y)} [C(x) \rightarrow C(y)]$$

such that with $1_{C(x)}$ being the function $u \mapsto u$ on $C(x)$ we have $f(x, x, \text{refl}(x)) = 1_{C(x)}$.

Foregoing extensional identity and using the induction principle encapsulated in **Id**-elimination and **Id**-equality in its stead, is crucial to the more subtle homotopy interpretations of type theory.

4 Constructive set theories

Constructive Set Theory was introduced by Myhill in a seminal paper [52], where a specific axiom system **CST** was introduced. Through developing constructive set theory he wanted to isolate the principles underlying Bishop's conception of what sets and functions are, and he wanted "these principles to be such as to make the process of formalization completely trivial, as it is in the classical case" ([52], p. 347). Myhill's **CST** was subsequently modified by Aczel and the resulting theory was called *Constructive Zermelo-Fraenkel set theory*, **CZF**. A hallmark of this theory is that it possesses a type-theoretic interpretation (cf. [2, 5]). Specifically, **CZF** has a scheme called Subset Collection Axiom (which is a generalization of Myhill's Exponentiation Axiom) whose formalization was directly inspired by the type-theoretic interpretation.

The language of **CZF** is the same first order language as that of classical Zermelo-Fraenkel Set Theory, **ZF** whose only non-logical symbol is \in . The logic of **CZF** is intuitionistic first order logic with equality. Among its non-logical axioms are *Extensionality*, *Pairing* and *Union* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize.

Infinity: $\exists x \forall u [u \in x \leftrightarrow (\emptyset = u \vee \exists v \in x \ u = v + 1)]$ where $v + 1 = v \cup \{v\}$.

Set Induction: $\forall x [\forall y \in x \phi(y) \rightarrow \phi(x)] \rightarrow \forall x \phi(x)$

Restricted or Bounded Separation: $\forall a \exists b \forall x [x \in b \leftrightarrow x \in a \wedge \phi(x)]$

for all *restricted* formulae ϕ . A set-theoretic formula is *restricted* if it is constructed from prime formulae using $\neg, \wedge, \vee, \rightarrow, \forall x \in y$ and $\exists x \in y$ only.

Strong Collection: For all formulae ϕ ,

$$\forall a [\forall x \in a \exists y \phi(x, y) \rightarrow \exists b [\forall x \in a \exists y \in b \phi(x, y) \wedge \forall y \in b \exists x \in a \phi(x, y)]] .$$

Subset Collection: For all formulae ψ ,

$$\begin{aligned} &\forall a \forall b \exists c \forall u [\forall x \in a \exists y \in b \psi(x, y, u) \rightarrow \\ &\quad \exists d \in c [\forall x \in a \exists y \in d \psi(x, y, u) \wedge \forall y \in d \exists x \in a \psi(x, y, u)]] . \end{aligned}$$

The Subset Collection schema easily qualifies as the most intricate axiom of **CZF**.

We shall also consider an additional axiom that holds true in the type-theoretic interpretation of Aczel if the type theory is equipped with W -types. To introduce it, we need the notion of a regular set. The formula in the language of **CZF** defining the property of a set A that it is *regular* states that A is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a \exists y (\langle x, y \rangle \in R)$, then there is a set $b \in A$ such that

$$\forall x \in a \exists y \in b (\langle x, y \rangle \in R) \wedge \forall y \in b \exists x \in a (\langle x, y \rangle \in R).$$

In particular, if $R : a \rightarrow A$ is a function, then the image of R is an element of A . Let $\text{Reg}(A)$ denote this assertion. With this auxiliary definition we can state the

Regular Extension Axiom **REA**

$$\forall x \exists y [x \subseteq y \wedge \text{Reg}(y)] .$$

There are several yet stronger forms of the axiom of choice that are validated in the type-theoretic interpretation (see [2]). Stating them requires some preliminary definitions.

4.1 The axiom of choice in constructive set theories

Among the axioms of set theory, the axiom of choice is distinguished by the fact that it is the only one that one finds mentioned in workaday mathematics. In the mathematical world of the beginning of the 20th century, discussions about the status of the axiom of choice were important. In 1904 Zermelo proved that every set can be well-ordered by employing the axiom of choice. While Zermelo argued that it was self-evident, it was also criticized as an excessively non-constructive principle by some of the most distinguished analysts of the day, notably Borel, Baire, and Lebesgue. At first blush this reaction against the axiom of choice utilized in Cantor's new theory of sets is surprising as the French analysts had used and continued to use choice principles routinely in their work. However, in the context of 19th century classical analysis only the Axiom of Dependent Choices, **DC**, is invoked and considered to be natural, while the full axiom of choice is unnecessary and even has some counterintuitive consequences.

Unsurprisingly, the axiom of choice does not have a unambiguous status in constructive mathematics either. On the one hand it is said to be an immediate consequence of the constructive interpretation of the quantifiers. Any proof of $\forall x \in A \exists y \in B \phi(x, y)$ must yield a function $f : A \rightarrow B$ such that $\forall x \in A \phi(x, f(x))$. This is certainly the case in Martin-Löf's intuitionistic theory of types. On the other hand, it has been observed that the full axiom of choice cannot be added to systems of extensional constructive set theory without yielding constructively unacceptable cases of excluded middle (see [19]). In extensional intuitionistic set theories, a proof of a statement $\forall x \in A \exists y \in B \phi(x, y)$, in general, provides only a function F , which when fed a proof p witnessing $x \in A$, yields $F(p) \in B$ and $\phi(x, F(p))$. Therefore, in the main, such an F cannot be rendered a function of x alone. Choice will then hold over sets which have a canonical proof function, where a constructive function h is a canonical proof function for A if for each $x \in A$, $h(x)$ is a constructive proof that $x \in A$. Such sets having natural canonical proof functions “built-in” have been called *bases* (cf. [91], p. 841).

Some constructive choice principles In many a text on constructive mathematics, axioms of countable choice and dependent choices are accepted as constructive principles. This is, for instance, the case in Bishop's constructive mathematics (cf. [12]) as well as Brouwer's intuitionistic analysis (cf. [91], Ch. 4, Sect. 2). Myhill also incorporated these axioms in his constructive set theory [52].

The weakest constructive choice principle we shall consider is the *Axiom of Countable Choice*, **AC** $_{\omega}$, i.e. whenever F is a function with domain ω such that $\forall i \in \omega \exists y \in F(i)$, then there exists a function f with domain ω such that $\forall i \in \omega f(i) \in F(i)$.

A mathematically very useful axiom to have in set theory is the *Dependent Choices Axiom*, **DC**, i.e., for all formulae ψ , whenever

$$(\forall x \in a) (\exists y \in a) \psi(x, y)$$

and $b_0 \in a$, then there exists a function $f : \omega \rightarrow a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega) \psi(f(n), f(n+1)).$$

Even more useful is the *Relativized Dependent Choices Axiom*, **RDC**. It asserts that for arbitrary formulae ϕ and ψ , whenever

$$\forall x [\phi(x) \rightarrow \exists y (\phi(y) \wedge \psi(x, y))]$$

and $\phi(b_0)$, then there exists a function f with domain ω such that $f(0) = b_0$ and

$$(\forall n \in \omega) [\phi(f(n)) \wedge \psi(f(n), f(n+1))].$$

In addition to the “traditional” axioms of choice stated above, the interpretation of set theory in type theory validates several new choice principles which are not well known. To state them we need to introduce various operations on classes.

Remark 4.1 Let **CZF** $_{Exp}$ denote the modification of **CZF** with Exponentiation in place of Subset Collection. In almost all the results of this paper, **CZF** could be replaced by **CZF** $_{Exp}$, that is to say, for the purposes of this paper it is enough to assume Exponentiation rather than Subset Collection. However, in what follows we shall not point this out again.

Definition 4.1 (CZF) If A is a set and B_x are classes for all $x \in A$, we define a class $\prod_{x \in A} B_x$ by:

$$\prod_{x \in A} B_x := \{f \mid f : A \rightarrow \bigcup_{x \in A} B_x \wedge \forall x \in A (f(x) \in B_x)\}. \quad (1)$$

If A is a class and B_x are classes for all $x \in A$, we define a class $\sum_{x \in A} B_x$ by:

$$\sum_{x \in A} B_x := \{\langle x, y \rangle \mid x \in A \wedge y \in B_x\}. \quad (2)$$

If A is a class and a, b are sets, we define a class $\mathbf{I}(A, a, b)$ by:

$$\mathbf{I}(A, a, b) := \{z \in 1 \mid a = b \wedge a, b \in A\}. \quad (3)$$

If A is a class and for each $a \in A$, B_a is a set, then

$$\mathbf{W}_{a \in A} B_a$$

is the smallest class Y such that whenever $a \in A$ and $f : B_a \rightarrow Y$, then $\langle a, f \rangle \in Y$.

Lemma 4.2 (CZF) If A, B, a, b are sets and B_x are sets for all $x \in A$, then $\prod_{x \in A} B_x$, $\sum_{x \in A} B_x$ and $\mathbf{I}(A, a, b)$ are sets.

Proof. [67, Lemma 2.5]. □

In the following we shall introduce several inductively defined classes, and, moreover, we have to ensure that such classes can be formalized in **CZF**.

We define an *inductive definition* to be a class of ordered pairs. If Φ is an inductive definition and $\langle x, a \rangle \in \Phi$ then we write

$$\frac{x}{a} \Phi$$

and call $\frac{x}{a} \Phi$ an (*inference*) *step* of Φ , with set x of *premisses* and *conclusion* a . For any class Y , let

$$\Gamma_\Phi(Y) = \{a \mid \exists x (x \subseteq Y \wedge \frac{x}{a} \Phi)\}.$$

The class Y is Φ -closed if $\Gamma_\Phi(Y) \subseteq Y$. Note that Γ is monotone; i.e. for classes Y_1, Y_2 , whenever $Y_1 \subseteq Y_2$, then $\Gamma(Y_1) \subseteq \Gamma(Y_2)$.

We define the class *inductively defined by* Φ to be the smallest Φ -closed class. The main result about inductively defined classes states that this class, denoted $\mathbf{I}(\Phi)$, always exists.

Lemma 4.3 (CZF) (Class Inductive Definition Theorem) *For any inductive definition Φ there is a smallest Φ -closed class $\mathbf{I}(\Phi)$.*

Proof. [2], section 4.2 or [4], Theorem 5.1. □

Lemma 4.4 (CZF + REA) *If A is a set and B_x is a set for all $x \in A$, then $\mathbf{W}_{a \in A} B_a$ is a set.*

Proof. This follows from [3], Corollary 5.3. □

Lemma 4.5 (CZF)

There exists a smallest $\Pi\Sigma$ -closed class, i.e., a smallest class \mathbf{Y} such that the following hold:

(i) $n \in \mathbf{Y}$ for all $n \in \omega$;

(ii) $\omega \in \mathbf{Y}$;

(iii) $\prod_{x \in A} B_x \in \mathbf{Y}$ and $\sum_{x \in A} B_x \in \mathbf{Y}$ whenever $A \in \mathbf{Y}$ and $B_x \in \mathbf{Y}$ for all $x \in A$.

Likewise, there exists a smallest $\Pi\Sigma\mathbf{I}$ -closed class, i.e. a smallest class \mathbf{Y}^ , which, in addition to the closure conditions (i)–(iii) above, satisfies:*

(iv) $\mathbf{I}(A, a, b) \in \mathbf{Y}^*$ whenever $A \in \mathbf{Y}^*$ and $a, b \in A$.

Proof. [67, Lemma 2.8]. □

Definition 4.2 The $\Pi\Sigma$ -generated sets are the sets in the smallest $\Pi\Sigma$ -closed class. Similarly one defines the $\Pi\Sigma\mathbf{I}$, $\Pi\Sigma\mathbf{W}$ and $\Pi\Sigma\mathbf{WI}$ -generated sets.

A set P is a *base* if for any P -indexed family $(X_a)_{a \in P}$ of inhabited sets X_a , there exists a function f with domain P such that, for all $a \in P$, $f(a) \in X_a$.

$\Pi\Sigma\text{--AC}$ is the statement that every $\Pi\Sigma$ -generated set is a base. Similarly one defines the axioms $\Pi\Sigma\mathbf{I}\text{--AC}$, $\Pi\Sigma\mathbf{WI}\text{--AC}$, and $\Pi\Sigma\mathbf{W}\text{--AC}$.

The *presentation axiom*, \mathbf{PAx} , states that every set is the surjective image of a base.

Lemma 4.3

- (i) $(\mathbf{CZF}) \Pi\Sigma\text{--AC}$ and $\Pi\Sigma\mathbf{I}\text{--AC}$ are equivalent.
- (ii) $(\mathbf{CZF} + \mathbf{REA}) \Pi\Sigma\mathbf{W}\text{--AC}$ and $\Pi\Sigma\mathbf{WI}\text{--AC}$ are equivalent.

Proof. [67, 2.12]. □

4.2 Large sets in constructive set theory

Large cardinals play a central role in modern set theory. This section deals with large cardinal properties in the context of intuitionistic set theories. Since in intuitionistic set theory \in is not a linear ordering on ordinals the notion of a cardinal does not play a central role. Consequently, one talks about “*large set properties*” instead of “*large cardinal properties*”. When stating these properties one has to proceed rather carefully. Classical equivalences of cardinal notion might no longer prevail in the intuitionistic setting, and one therefore wants to choose a rendering which intuitionistically retains the most strength. On the other hand certain notions have to be avoided so as not to imply excluded third. To give an example, cardinal notions like measurability, supercompactness and hugeness have to be expressed in terms of elementary embeddings rather than ultrafilters.

We shall, however, not concern ourselves with very large cardinals here and rather restrict attention to the very first notions of largeness introduced by Hausdorff and Mahlo, that is, inaccessible and Mahlo sets and the pertaining hierarchies of inaccessible and Mahlo sets.

We have already seen one notion of largeness, namely that of a regular set. In \mathbf{ZFC} , a regular set which itself is a model of the axioms of \mathbf{CZF} is of the form V_κ with κ a strongly inaccessible cardinal.⁵ In the context of \mathbf{CZF} this notion is much weaker.

Definition 4.4 If A is a transitive set and ϕ is a formula with parameters in A we denote by ϕ^A the formula which arises from ϕ by replacing all unbounded quantifiers $\forall u$ and $\exists v$ in ϕ by $\forall u \in A$ and $\exists v \in A$, respectively.

We can view any transitive set A as a structure equipped with the binary relation $\in_A = \{\langle x, y \rangle \mid x \in y \in A\}$. A set-theoretic sentence whose parameters lie in A , then has a canonical interpretation in (A, \in_A) by interpreting \in as \in_A , and $(A, \in_A) \models \phi$ is logically equivalent to ϕ^A . We shall usually write $A \models \phi$ in place of ϕ^A .

A set I is said to be *weakly inaccessible* if I is a regular set such that $I \models \mathbf{CZF}$.

The strong regular extension axiom, \mathbf{sREA} , states that every set is an element of a weakly inaccessible set.

There is a more ‘algebraic’ way of expressing weak inaccessibility. Stating it requires some definitions.

Definition 4.5 For sets A, B we denote by $\mathbf{mv}^A(B)$ the collection of all full relations from A to B , i.e., of those relations $R \subseteq A \times B$ such that $\forall x \in A \exists y \in B \langle x, y \rangle \in R$. A set C is said to be *full in* $\mathbf{mv}^A(B)$ if for all $R \in \mathbf{mv}^A(B)$ there exists $R' \in \mathbf{mv}^A(B)$ such that $R' \subseteq R$ and $R' \in C$.

For a set A , define $\bigwedge A$ to be the set $\{x \in 1 \mid \forall u \in A x \in u\}$, where $1 = \{\emptyset\}$.

⁵Note that \mathbf{CZF} with classical logic is the same theory as \mathbf{ZF} .

Proposition 4.6 *A set I is weakly inaccessible if I is a regular set such that the following are satisfied:*

1. $\omega \in I$,
2. $\forall a \in I \bigcup a \in I$,
3. $\forall a \in I [a \text{ inhabited} \rightarrow \bigcap a \in I]$,
4. $\forall A, B \in I \exists C \in I \quad C \text{ is full in } \mathbf{mv}({}^A B)$.

Proof: [5, 10.26].

We will consider two stronger notions.

Definition 4.7 A set I is called *inaccessible* if I is weakly inaccessible and for all $x \in I$ there exists a regular set $y \in I$ such that $x \in y$.

A set M is said to be *Mahlo* if M is inaccessible and for every $R \in \mathbf{mv}({}^M M)$ there exists an inaccessible $I \in M$ such that

$$\forall x \in I \exists y \in I \langle x, y \rangle \in R.$$

4.3 Fragments of second order arithmetic

The proof-theoretic strength of theories is commonly calibrated using standard theories and their canonical fragments. In classical set theory this linear line of consistency strengths is couched in terms of large cardinal axioms while for weaker theories the line of reference systems traditionally consist of subsystems of second order arithmetic. The observation that large chunks of mathematics can already be formalized in fragments of second order arithmetic goes back to Hilbert and Bernays [32], and has led to a systematic research program known as *Reverse Mathematics*. Below we give an account of the syntax of \mathcal{L}_2 and frequently considered axiomatic principles.

Definition 4.6 The language \mathcal{L}_2 of second-order arithmetic contains number variables x, y, z, u, \dots , set variables $X, Y, Z, U, V, A, B, C, \dots$ (ranging over subsets of \mathbb{N}), the constant 0, function symbols $Suc, +, \cdot$, and relation symbols $=, <, \in$. Suc stands for the successor function. *Terms* are built up as usual. For $n \in \mathbb{N}$, let \bar{n} be the canonical term denoting n . Formulae are built from the prime formulae $s = t$, $s < t$, and $s \in X$ using $\wedge, \vee, \neg, \forall x, \exists x, \forall X$ and $\exists X$ where s, t are terms. Note that equality in \mathcal{L}_2 is only a relation on numbers. However, equality of sets will be considered a defined notion, namely $X = Y$ if and only if $\forall x [x \in X \leftrightarrow x \in Y]$. As per usual, number quantifiers are called bounded if they occur in the context $\forall x (x < s \rightarrow \dots)$ or $\exists x (x < s \wedge \dots)$ for a term s which does not contain x . The Σ_0^0 -formulae are those formulae in which all quantifiers are bounded number quantifiers. For $k > 0$, Σ_k^0 -formulae are formulae of the form $\exists x_1 \forall x_2 \dots Q x_k \phi$, where ϕ is Σ_0^0 ; Π_k^0 -formulae are those of the form $\forall x_1 \exists x_2 \dots Q x_k \phi$. The union of all Π_k^0 - and Σ_k^0 -formulae for all $k \in \mathbb{N}$ is the class of *arithmetical* or Π_∞^0 -formulae. The Σ_k^1 -formulae (Π_k^1 -formulae) are the formulae $\exists X_1 \forall X_2 \dots Q X_k \phi$ (resp. $\forall X_1 \exists X_2 \dots Q X_k \phi$) for arithmetical ϕ .

The basic axioms in all theories of second-order arithmetic are the defining axioms of 0, 1, +, \cdot , < and the *induction axiom*

$$\forall X (0 \in X \wedge \forall x (x \in X \rightarrow x + 1 \in X) \rightarrow \forall x (x \in X)),$$

respectively the *scheme of induction*

$$\mathbf{IND} \quad \phi(0) \wedge \forall x (\phi(x) \rightarrow \phi(x + 1)) \rightarrow \forall x \phi(x),$$

where ϕ is an arbitrary \mathcal{L}_2 -formula. We consider the axiom scheme of \mathcal{C} -comprehension for formula classes \mathcal{C} which is given by

$$\mathbf{C-CA} \quad \exists X \forall u (u \in X \leftrightarrow \phi(u))$$

for all formulae $\phi \in \mathcal{C}$ (of course, X must not be free in ϕ).

For each axiom scheme **Ax** we denote by (\mathbf{Ax}) the theory consisting of the basic arithmetical axioms, the scheme $\Pi_\infty^0\text{-CA}$, the scheme of induction and the scheme **Ax**. If we replace the scheme of induction by the

induction axiom, we denote the resulting theory by $(\mathbf{Ax})_0$. An example for these notations is the theory $(\Pi_1^1\text{-}\mathbf{CA})$ which contains the induction scheme, whereas $(\Pi_1^1\text{-}\mathbf{CA})_0$ only contains the induction axiom in addition to the comprehension scheme for Π_1^1 -formulae.

In the basic system one can introduce defined symbols for all primitive recursive functions. Especially, let $\langle, \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a primitive recursive and bijective pairing function. The x^{th} section of U is defined by $U_x := \{y : \langle x, y \rangle \in U\}$. Observe that a set U is uniquely determined by its sections on account of \langle, \rangle 's bijectivity. Any set R gives rise to a binary relation \prec_R defined by $y \prec_R x := \langle y, x \rangle \in R$. Using this coding we can formulate *C-axiom of choice* scheme for formula classes \mathcal{C} which is given by

$$\mathcal{C}\text{-}\mathbf{AC} \quad \forall x \exists Y \psi(x, Y) \rightarrow \exists Z \forall u \psi(x, Z_x),$$

for all formulae $\psi \in \mathcal{C}$ (Z must not be free in ψ).

Another important principle is *Bar induction*:

$$\mathbf{BI} \quad \forall X [\mathbf{WF}(\prec_X) \wedge \forall u (\forall v \prec_X u \phi(v) \rightarrow \phi(u)) \rightarrow \forall u \phi(u)]$$

for all formulae ϕ , where $\mathbf{WF}(\prec_X)$ expresses that \prec_X is well-founded, i.e., $\mathbf{WF}(\prec_X)$ stands for the formula

$$\forall Y [\forall u ((\forall v \prec_X u v \in Y) \rightarrow u \in Y) \rightarrow \forall u u \in Y].$$

Universes in type theory (with W -types) bear a strong relation to β -models which are models of the language of \mathcal{L}_2 or set theory for which the notion well-foundedness is absolute.

Definition 4.8 Any set A of natural numbers gives rise to a set $\mathfrak{X}_A := \{A_i \mid i \in \mathbb{N}\}$ of sets of natural numbers. A is said to be a β -model if the \mathcal{L}_2 -structure

$$\mathfrak{A} := (\mathbb{N}, \mathfrak{X}_A, 0, 1, +, \cdot, \in)$$

is a β -model, i.e., $\mathfrak{A} \models \Pi_\infty^0\text{-}\mathbf{CA}$, and whenever $Y \in \mathfrak{X}_A$ and $\mathfrak{A} \models \mathbf{WF}(\prec_Y)$ then \prec_Y is well-founded.

Obviously, the notion, the notion of β -model can be expressed in \mathcal{L}_2 .

An intuitionistic \mathcal{L}_2 -theory. There is an interesting version of second order arithmetic, which will be used in theory reductions, that classically has the same strength as full second order arithmetic, $(\Pi_\infty^1\text{-}\mathbf{CA})$, but when based on intuitionistic logic is of the same strength as \mathbf{T}_0^i .

Definition 4.9 **IARI** is a theory in the language of second order arithmetic. The logical rules of **IARI** are those of intuitionistic second order arithmetic. In addition to the usual axioms for intuitionistic second order logic, axioms are (the universal closures of):

1. **Induction:**

$$\phi(0) \wedge \forall n [\phi(n) \rightarrow \phi(n+1)] \rightarrow \forall n \phi(n)$$

for all formulae ϕ .

2. **Arithmetic Comprehension Schema:**

$$\exists X \forall n [n \in X \leftrightarrow \psi(x)]$$

for ψ arithmetical (parameters allowed).

3. **Replacement:**

$$\forall X [\forall n \in X \exists ! Y \phi(n, Y) \rightarrow \exists Z \forall n \in X \phi(n, Z_n)]$$

for all formulas ϕ . Here $\phi(n, Z_n)$ arises from $\phi(n, Z)$ by replacing each occurrence $t \in Z$ in the formula by $\langle n, t \rangle \in Z$.

4. Inductive Generation:

$$\forall U \forall X \exists Y [\mathbf{WP}_U(X, Y) \wedge (\forall n [\forall k (k \prec_X n \rightarrow \phi(k)) \rightarrow \phi(n)] \rightarrow \forall m \in Y \phi(m))],$$

for all formulas ϕ , where $k \prec_X n$ abbreviates $\langle k, n \rangle \in X$ and $\mathbf{WP}_U(X, Y)$ stands for

$$\mathbf{Prog}_U(X, Y) \wedge \forall Z [\mathbf{Prog}_U(X, Z) \rightarrow Y \subseteq Z]$$

with $\mathbf{Prog}_U(X, Y)$ being $\forall n \in U [\forall k (k \prec_X n \rightarrow k \in Y) \rightarrow n \in Y]$.

Remark 4.10 (IARI) Note that $\mathbf{WP}_U(X, Y)$ and $\mathbf{WP}_U(X, Y')$ imply $Y = Y'$, i.e. $\forall n (n \in Y \leftrightarrow n \in Y')$. Therefore, if $\mathbf{WP}_U(X, Y)$, then

$$\forall n \in U [\forall k \prec_X n \phi(k) \rightarrow \phi(n)] \rightarrow \forall m \in Y \phi(m)$$

holds for all formulae ϕ .

The latter principle will be referred to as “*induction over the well-founded part of \prec_X* ”. In the rest of this section we shall write $\mathbf{WF}(U, X)$ for the (extensionally) uniquely determined Y which satisfies $\mathbf{WP}_U(X, Y)$. The main tool for performing the well-ordering proof of [37] in **IARI** is the following principle of transfinite recursion.

Proposition 4.11 (IARI) *If $\mathbf{WP}_U(X, Y)$ and $\forall n \in Y \forall W \exists! V \psi(n, W, V)$, then there exists Z such that*

$$\forall n \in Y \psi(n, \bigcup \{(Z)_k : k \prec_X n\}, (Z)_n).$$

Proof: See [59, 6.4]. □

5 On relating theories I

The first result relates intuitionistic explicit mathematics to constructive set theory and a fragment of **MLTT**. Let $\mathbf{MLT}_{1W}V$ be the fragment of **MLTT** with only one universe \mathcal{U}_0 where the W -constructor can solely be applied to families of types in \mathcal{U}_0 but one can also form the type $V := W_{(A:\mathcal{U}_0)}A$ (something that could be called the type of Brouwer ordinals of \mathcal{U}_0). We shall also consider the type theory \mathbf{MLT}_{1W} which is the fragment of $\mathbf{MLT}_{1W}V$ without the type V .

A principle of omniscience. Certain basic principles of classical mathematics are taboo for the constructive mathematician. Bishop called them *principles of omniscience*. The limited principle of omniscience, **LPO**, is an instance of the law of excluded middle which usually serves as a line of demarcation, separating “constructive” from “non-constructive” theories. In the case of **CZF**, adding the law of excluded middle even just for atomic statements of the form $a \in b$ results in an enormous increase in proof strength, pushing it up beyond that of Zermelo set theory. However, **LPO** can be added to **CZF** without affecting its proof-theoretic strength. **LPO** has the pleasant side effect that one can carry out elementary analysis pretty much in the same way as in any standard text book.

Definition 5.1 Let $2^{\mathbb{N}}$ be Cantor space, i.e the set of all functions from the naturals into $\{0, 1\}$. Limited Principle of Omniscience (**LPO**):

$$\forall f \in 2^{\mathbb{N}} [\exists n f(n) = 1 \vee \forall n f(n) = 0].$$

Theorem 5.2 *The following theories have the same proof-theoretic strength:*

- (i) *Intuitionistic explicit mathematics, \mathbf{T}_0^i .*

- (ii) *Constructive Zermelo-Fraenkel set theory with the Regular Extension Axiom, **CZF** + **REA**.*
- (iii) *Constructive Zermelo-Fraenkel set theory with the strong Regular Extension Axiom and **RDC**, **CZF** + **sREA** + **RDC**.*
- (iv) **CZF** + **REA** + **$\Pi\Sigma W$ -AC** + **RDC** + **PAx**.
- (v) *The extensional type theory $\mathbf{MLT}_{1W}^{\text{ext}} \mathbf{V}$.*
- (vi) $\mathbf{MLT}_{1W} \mathbf{V}$.
- (vii) *The extensional type theory $\mathbf{MLT}_{1W}^{\text{ext}}$.*
- (viii) \mathbf{MLT}_{1W} .
- (ix) *The classical subsystem of second order arithmetic $(\Sigma_2^1\text{-AC}) + \mathbf{BI}$ (same as $(\Delta_2^1\text{-CA}) + \mathbf{BI}$).*
- (x) *The intuitionistic system **IARI** of second order arithmetic.*
- (xi) *Classical Kripke-Platek set theory **KP** plus the axiom asserting that every set is contained in an admissible set. (This theory is often denoted by **KPi**.)*
- (xii) **CZF** + **REA** + **RDC** + **LPO**.

Proof: The equivalence of (i),(ii),(iii),(iv),(v),(vi),(vii),(viii),(ix),(x), and (xi) follows from [59], Theorem 3.9, Proposition 5.3, Theorem 5.13 and Theorem 6.13 plus the extra observation that the interpretation of **IRA** in $\mathbf{MLT}_{1W}^{\text{ext}}$ defined in [59, Definition 6.5] and proved to be an interpretation in [59, Theorem 6.9] actually only requires the intensional identity type. It was already observed by Palmgren [55] that the interpretations of theories of iterated, strictly positive inductive definitions in type theory works with the intensional identity, and the same argument applies here.

The equivalence of (ii) and (iii) follows from [63, Theorem 4.7], where the principle **sREA** is denoted by **INAC**.

That the theory **CZF** + **REA** + **RDC** + **LPO** has a realizability interpretation in $(\Sigma_2^1\text{-AC}) + \mathbf{BI}$ follows by an extension of the techniques used in [77, Theorem 6.2]. The proof furnished a realizability model for **CZF** + **RDC** + **LPO** that is based on recursion in the type-2 object $E : (N \rightarrow N) \rightarrow N$ with $E(f) = n + 1$ if $f(n) = 0$ and $\forall i < n f(i) > 0$ and $E(f) = 0$ if $\forall n f(n) > 0$. Recursion in E is formalizable in the theory of bar induction, i.e. $(\Pi_\infty^0\text{-CA}) + \mathbf{BI}$, which is known to have the same strength as **CZF** (see [77, Theorem 2.2]). The same recursion theory (or partial combinatory algebra) can be employed in extending the modeling of a type structure given in [77, §5] to the larger type structure needed for **CZF** + **REA** + **RDC** + **LPO**. This is achieved by basically taking the type structure in [59, 5.8] but changing the underlying partial combinatory algebra to the one obtained from recursion in the type two object E rather than the usual one provided by the partial recursive functions on \mathbb{N} .

It is very likely that the interpretation also validates **$\Pi\Sigma W$ -AC** and **PAx**, but this hasn't yet been checked. At any rate, we have shown the proof-theoretic equivalence of all theories. \square

The foregoing proof establishes the claimed results, however, we'd like to look at Theorem 5.2 in more detail, especially at its proof(s) and the information one can extract from it.

For starters, what does the phrase “same proof-theoretic strength” mean? At a minimum it means that the theories ought to be finitistically equiconsistent. Here it means that they prove at least the same Π_2^0 statements of the language of first-order arithmetic. But more can be shown. A result we will be working toward is that many of the intuitionistic theories of Theorem 5.2 prove the same arithmetical statements. In particular it will be shown that the extensional and the intensional type theories prove the same arithmetical statements. An arithmetical statement gives rise to a type via the propositions-as-types paradigm, so by conservativity of one type theory over another with respect to arithmetic statements we mean that the same arithmetical types are provably inhabited in both theories.

The question of the relation between intensional and extensional type theories has been addressed before by Hofmann in [35]. The set-up there, though, is somewhat different in that the intensional type theory TT_I of [35] is not a pure intensional type theory. It has two extensional rules called *functional extensionality* and *uniqueness of identity*:

$$\begin{array}{lcl}
\text{(ID-UNI-I)} & & \frac{\Gamma \vdash s : A \quad \Gamma \vdash p : \text{Id}(A, s, s)}{\Gamma \vdash \text{IdUni}(A, s, p) : \text{Id}(\text{Id}(A, s, s), p, \text{refl}(s))} \\
\text{(EXT-FORM)} & & \frac{\Gamma \vdash f, g : \Pi_{(x:A)} B(x) \quad \Gamma, x : A \vdash p(x) : \text{Id}(B(x), fx, gx)}{\Gamma \vdash \text{Ext}(f, g, p) : \text{Id}(\Pi_{(x:A)} B(x), f, g)}.
\end{array}$$

These rules are not provable in the purely intensional context, so as a result, we are pursuing a different question here.

Proposition 5.3 \mathbf{T}_0^i can be interpreted in $\mathbf{CZF} + \mathbf{REA}$. The interpretation preserves (at least) all arithmetic statements.

Proof: The proof of [59] Theorem 3.9 provides an interpretation of \mathbf{T}_0^i in $\mathbf{CZF} + \mathbf{REA}$ which is essentially a class model of \mathbf{T}_0^i inside $\mathbf{CZF} + \mathbf{REA}$. Having defined an applicative structure, the classifications are defined inductively along the (intuitionistic) ordinals. This is inspired by Feferman's construction of a model of \mathbf{T}_0 in [21, Theorem 4.1.1]. Inspection of the translation confirms that arithmetic statements get preserved. \square

Proposition 5.4 (i) $\mathbf{CZF} + \mathbf{REA}$ has an interpretation in $\mathbf{MLT}_{1W}V$.

(ii) $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma W\text{-AC} + \mathbf{RDC} + \mathbf{PAx}$ has an interpretation in $\mathbf{MLT}_{1W}^{ext}V$.

Proof: (i) and (ii) follow from [3]. The interpretation uses the type V and two propositional functions

$$\begin{aligned}
& \doteq : V \times V \rightarrow \mathcal{U}_0 \\
& \dot{\in} : V \times V \rightarrow \mathcal{U}_0
\end{aligned}$$

to interpret $=$ and \in . For (i), the identity type does not play any role. For (ii) one needs the extensionality of function types. \square

Proposition 5.5 $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma W\text{-AC} + \mathbf{RDC} + \mathbf{PAx}$ is conservative over $\mathbf{CZF} + \mathbf{REA} + \mathbf{FT-AC}$ for statements of finite type arithmetic (i.e., of the language of \mathbf{HA}^ω).

Proof: From [67, Theorem 5.23] it follows that $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma W\text{-AC} + \mathbf{RDC} + \mathbf{PAx}$ and $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma W\text{-AC}$ prove the same sentences of finite type arithmetic (and more) since the inner model $H(Y_W^*)$ satisfies $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma W\text{-AC} + \mathbf{RDC} + \mathbf{PAx}$, assuming $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma W\text{-AC}$ in the background.

By [66, Theorem 4.33], there is an interpretation of $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma W\text{-AC}$ in $\mathbf{CZF} + \mathbf{REA}$. Inspection shows that, in the presence of $\mathbf{FT-AC}$, the meanings of statements of finite type arithmetic are preserved under this interpretation. \square

Proposition 5.6 For θ a sentence of arithmetic let $\|\theta\|$ be the corresponding type term according to the propositions-as-types translation. If

$$\mathbf{MLT}_{1W}^{ext}V \vdash t : \|\theta\|$$

for some term t , then

$$\mathbf{CZF} + \mathbf{REA} + \mathbf{FT-AC} \vdash \theta_{set}$$

with θ_{set} denoting the standard set-theoretic rendering of θ .

Proof: Assume $\mathbf{MLT}_{1W}^{\text{ext}} \mathbf{V} \vdash t : \|\theta\|$. The interpretation $^{\wedge}$ of $\mathbf{MLT}_{1W}^{\text{ext}} \mathbf{V}$ into $\mathbf{CZF} + \mathbf{REA}$ given in [67, §6] yields $\mathbf{CZF} + \mathbf{REA} \vdash (t : \|\theta\|)^{\wedge}$. Inspection shows that $(t : \|\theta\|)^{\wedge}$ is a statement about the finite type structure over ω . One then sees, with the help of $\mathbf{FT-AC}$, that θ_{set} holds. This is similar to the proof of [67, Theorem 3.15]. \square

Theorem 5.7 $\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma\mathbf{W-AC} + \mathbf{RDC} + \mathbf{PAx}$ is conservative over

$$\mathbf{IKP} + \forall x \exists y [x \in y \wedge y \text{ is an admissible set}]$$

for arithmetic statements.

Proof: We shall use the shorthand \mathbf{IKPi} for the latter theory. By Proposition 5.5 it suffices to show that $\mathbf{CZF} + \mathbf{REA} + \mathbf{FT-AC}$ is conservative over \mathbf{IKPi} for arithmetic statements. [59, Theorem 5.11] shows that $\mathbf{MLT}_{1W}^{\text{ext}} \mathbf{V}$ has an interpretation in the classical theory \mathbf{KPi} where types are interpreted as subsets of ω and crucially dependent products of types are interpreted as sets of indices of partial recursive functions. This also furnishes an interpretation of $\mathbf{CZF} + \mathbf{REA} + \mathbf{FT-AC}$ in \mathbf{KPi} since the former is interpretable in $\mathbf{MLT}_{1W}^{\text{ext}} \mathbf{V}$. The interpretation also works for \mathbf{IKPi} as definition by (transfinite) Σ -recursion works in intuitionistic \mathbf{KP} as well (see [4, Sec. 11] and [5, Sec. 19]). The inductive definition of 5.8 in [59] proceeds along the ordinals and focusses on successor ordinals, seemingly requiring a classical case distinction as to whether an ordinal is a successor or a limit or 0, but this is actually completely irrelevant.

Now, the upshot of this hereditarily recursive interpretation is that every Π_2^0 theorem of $\mathbf{CZF} + \mathbf{REA} + \mathbf{FT-AC}$ is provable in \mathbf{IKPi} . To be able to extend this approach to all of arithmetic, one needs a more abstract type structure such that interpretability entails deducibility. The conservativity of $\mathbf{HA}^\omega + \mathbf{FT-AC}$ over \mathbf{HA} , due to Goodman [28, 29], provides the template. The two steps of Goodman's second proof have been neatly separated by Beeson [10] to construct a general methodology for showing an intuitionistic theory T to be conservative over another theory S for arithmetic statements. The idea is to combine two interpretations, where the first uses functions that are recursive relative to a generic oracle and the second step is a forcing construction. The same idea has been used by Gordeev [30], and in more recent times in [15, 17], establishing several conservativity results.

The oracle \mathcal{O} will be a fixed but arbitrary partial function from \mathbb{N} to $\{0, 1\}$. A partial function ϕ is recursive relative to \mathcal{O} if it is given by a Turing machine with access to \mathcal{O} . During a computation the oracle may be consulted about the value of $\mathcal{O}(n)$ for several n . If $\mathcal{O}(n)$ is defined it will return that value and the computation will continue, but if $\mathcal{O}(n)$ is not defined no response will be coming forward and the computation will never come to a halt. The idea of the second interpretation step is that on account of \mathcal{O} 's arbitrariness it can be interpreted in many ways. Given an arithmetic statement θ an oracle \mathcal{O}_θ can be engineered so that in a forcing model realizability of θ with functions computable relative to \mathcal{O}_θ entails the truth of θ . The final step, then, is achieved by noticing that for arithmetic statements forceability (where the forcing conditions are finite partial functions on \mathbb{N}) and validity coincide. For details we'll have to refer to [15, 17]. \square

Definition 5.8 Below we shall speak about arithmetical statements in various theories with differing languages. There is a canonical translation of the language of first and second order arithmetic into the language of set theory. However, it is perhaps less obvious what arithmetical statements mean in the context of type theory.

The terms of the language of \mathbf{HA} are to be translated in an obvious way, crucially using the type-theoretic recursor for the type \mathbb{N} . In this way each term t of \mathbf{HA} gets assigned a raw term \hat{t} of type theory. For details see [49, pp. 71–75], [9, XI.17] [91, Ch. 11, Sect. 2]. An equation $s = t$ of the language \mathbf{HA} is translated as a type-expression $\text{Id}(\mathbb{N}, \hat{s}, \hat{t})$. For complex formulas the translation proceeds in the obvious way.

We then say that two type theories TT_1 and TT_2 prove the same arithmetical statements if for all sentences A of \mathbf{HA} ,

$$TT_1 \vdash p : \hat{A} \text{ for some } p \quad \text{iff} \quad TT_2 \vdash p' : \hat{A} \text{ for some } p',$$

where \hat{A} denotes the type-theoretic translation of A .

Theorem 5.9 *The following theories prove the same arithmetical statements, i.e. statements of intuitionistic arithmetic:*

- (i) \mathbf{T}_0^i .
- (ii) $\mathbf{CZF} + \mathbf{REA}$.
- (iii) $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W-AC} + \mathbf{RDC} + \mathbf{PAx}$.
- (iv) $\mathbf{MLT}_{1W}^{ext}\mathbf{V}$.
- (v) $\mathbf{MLT}_{1W}\mathbf{V}$.
- (vi) \mathbf{MLT}_{1W}^{ext} .
- (vii) \mathbf{MLT}_{1W} .
- (viii) \mathbf{IARI} .

Proof: Let θ be an arithmetic sentence. Then we have

$$\begin{aligned}
\mathbf{T}_0^i \vdash \theta &\Rightarrow \mathbf{CZF} + \mathbf{REA} \vdash \theta \\
&\Rightarrow \mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W-AC} + \mathbf{RDC} + \mathbf{PAx} \vdash \theta \\
&\Rightarrow \mathbf{IKPi} \vdash \theta
\end{aligned}$$

by Proposition 5.3 and Theorem 5.7. Now it follows from Jäger's article [37] and from [39] that every initial segment of the proof-theoretic ordinal of \mathbf{IKPi} is provably well-founded in \mathbf{T}_0^i , and thus, if $\mathbf{IKPi} \vdash \theta$, then \mathbf{T}_0^i is sufficient to show that there is an infinite intuitionistic cut-free proof of θ . By induction on the length of the proof it then follows that all sequents in the proof are true, yielding that $\mathbf{T}_0^i \vdash \theta$. The upshot is that the theories of (i), (ii) and (iii) prove the same arithmetical statements. Furthermore, if $\mathbf{MLT}_{1W}^{ext}\mathbf{V} \vdash t : \|\theta\|$ for some term t , then $\mathbf{CZF} + \mathbf{REA} \vdash \theta$ by Proposition 5.6 and hence $\mathbf{T}_0^i \vdash \theta$.

So to finish the proof it would suffice to show that $\mathbf{T}_0^i \vdash \theta$ yields $\mathbf{MLT}_{1W}\mathbf{V} \vdash s : \|\theta\|$ for some term s . Now [59, Sec. 6] shows that the intuitionistic theory \mathbf{IARI} has the same proof-theoretic ordinal as \mathbf{IKPi} and \mathbf{T}_0^i . So from $\mathbf{T}_0^i \vdash \theta$ it follows that $\mathbf{IARI} \vdash \theta$. By [59, Theorem 6.9] we then get $\mathbf{MLT}_{1W}\mathbf{V} \vdash s : \|\theta\|$ for some term s , completing the circle. \square

Remark 5.10 Ordinal analysis played a crucial role in the proofs of Theorems 5.2 and 5.9. Having the same proof-theoretic ordinal allowed us to infer that \mathbf{T}_0^i , \mathbf{IKPi} and \mathbf{IARI} prove the same arithmetic statements. For a long time [37] was also the only proof that enabled one to reduce the classical theories $(\Delta_2^1\text{-CA}) + \mathbf{BI}$ and \mathbf{KPi} to classical \mathbf{T}_0 . There is now also a proof by Sato [80] for the reductions in the *classical* case that avoids proof-theoretic ordinals.

6 On relating theories II: MLTT and friends

So far we have only gathered results concerning theories that are of the strength of Martin-Löf type theory with one universe. The earlier quote by Harris speculated on the strength of type theory with infinitely many universes. As it turns out, similar techniques can be applied in this context as well.

To begin with, we shall define versions of explicit mathematics, second order arithmetic and constructive set theory featuring analogues of universes.

6.1 \mathbf{T}_0^i with universes.

Definition 6.1 Systems of explicit mathematics with universes have been defined and studied in several papers (cf. [40, 41, 42]) and were probably first introduced by Feferman [24].

By $\mathbf{T}_0^i + \bigcup_n \mathbf{U}_n$ we denote an extension of \mathbf{T}_0^i whose language has infinitely many classification constants $\mathbf{U}_0, \mathbf{U}_1, \dots$ and the following axioms for each constant \mathbf{U}_n .

1. $\mathbf{N} \varepsilon \mathbf{U}_n$ and $\mathbf{U}_i \varepsilon \mathbf{U}_n$ for $i < n$.
2. $\forall x \varepsilon \mathbf{U}_n \exists X x = X$ (i.e. every element of \mathbf{U}_n is a classification).
3. For every elementary formula $\psi(x, \vec{v}, X_1, \dots, X_r)$ with all classification variables exhibited and which does not contain constants \mathbf{U}_i with $i \geq n$,

$$\forall X_1, \dots, X_r \varepsilon \mathbf{U}_n \exists Y [Y \varepsilon \mathbf{U}_n \wedge Y \simeq \{x : \psi(x, \vec{v}, X_1, \dots, X_r)\}].$$

4. $\forall X \varepsilon \mathbf{U}_n [\forall x \varepsilon X \exists Y \varepsilon \mathbf{U}_n fx \simeq Y \rightarrow \exists Z [Z \in \mathbf{U}_n \wedge Z \simeq \mathbf{j}(X, f)]]$.
5. $\forall X, Y \varepsilon \mathbf{U}_n \exists Z [Z \in \mathbf{U}_n \wedge Z \simeq \mathbf{i}(X, Y)]$.

In other words, a classification \mathbf{U}_n is a universe containing $\mathbf{N}, \mathbf{U}_0, \dots, \mathbf{U}_{n-1}$ closed under elementary comprehension, join and inductive generation.

By $\mathbf{T}_0^i + \bigcup_{i < n} \mathbf{U}_i$ we denote the theory with just the universes $\mathbf{U}_0, \dots, \mathbf{U}_{n-1}$ and their pertaining axioms.

6.2 Universes in intuitionistic second order arithmetic.

It is also useful to have a many universes version of **IARI** to obtain an intuitionistic theory of second order arithmetic which can be easily interpreted in **MLTT**. One idea would be to adopt the notion of β -model from Definition 4.8 to serve as a notion of universe. However, a β -model comes with an explicit countable enumeration of its sets and therefore it would be difficult if not impossible to model such structures in **MLTT**. Instead, an option is to add set predicates $\mathfrak{U}_0, \mathfrak{U}_1, \dots$ to the language \mathcal{L}_2 that are intended to apply to sets of natural numbers with the aim of singling out collections of sets that have universe-like properties.

Definition 6.2 The theory **IARI** + $\bigcup_n \mathfrak{U}_n$ has additional predicates $\mathfrak{U}_0, \mathfrak{U}_1, \dots$ for creating new atomic formulas $\mathfrak{U}_n(X)$ ($n \in \mathbb{N}$), where X is a second order variable. We use abbreviations like $\forall X \in \mathfrak{U}_n \varphi$ and $\exists X \in \mathfrak{U}_n \varphi$ for $\forall X (\mathfrak{U}_n(X) \rightarrow \varphi)$ and $\exists X (\mathfrak{U}_n(X) \wedge \varphi)$, respectively. If ψ is any formula of this language, then $\psi^{\mathfrak{U}_n}$ arises from ψ by relativizing all second order quantifiers to \mathfrak{U}_n , i.e., replacing all quantifiers QX in ψ by $QX \in \mathfrak{U}_n$.

In addition to the axioms of **IARI** there are the following pertaining to the new predicates.

1. The predicates \mathfrak{U}_n are cumulative, i.e. $\forall X [\mathfrak{U}_i(X) \rightarrow \mathfrak{U}_j(X)]$ whenever $i \leq j$.
2. **Induction:**

$$\phi(0) \wedge \forall u [\phi(u) \rightarrow \phi(u+1)] \rightarrow \forall u \phi(u)$$

for all formulae ϕ .

3. **Arithmetic Comprehension Schema for \mathfrak{U}_n :**

$$Y_1, \dots, Y_r \in \mathfrak{U}_n \rightarrow \exists X \in \mathfrak{U}_n \forall u [u \in X \leftrightarrow \psi(u, Y_1, \dots, Y_r)]$$

if $\psi(u, Y_1, \dots, Y_r)$ is a formula with all free second order variables exhibited, in which all second order quantifiers are of the form $QX \in \mathfrak{U}_i$ for some $i < n$, and moreover, no predicates \mathfrak{U}_j for $j \geq n$ occur in it.

4. Replacement:

$$\forall X \in \mathfrak{U}_n [\forall u \in X \exists ! Y \in \mathfrak{U}_n \phi(u, Y) \rightarrow \exists Z \in \mathfrak{U}_n \forall u \in X \phi(u, Z_u)]$$

for all formulas ϕ . Here $\phi(u, Z_u)$ arises from $\phi(u, Z)$ by replacing each occurrence $t \in Z$ in the formula by $\langle u, t \rangle \in Z$.

5. Inductive Generation:

$$\forall U \in \mathfrak{U}_n \forall X \in \mathfrak{U}_n \exists Y \in \mathfrak{U}_n [\mathbf{WP}_U(X, Y) \wedge (\forall u [\forall v (v \prec_X u \rightarrow \phi(v)) \rightarrow \phi(u)] \rightarrow \forall x \in Y \phi(x))],$$

for all formulas ϕ , where $v \prec_X u$ abbreviates $\langle v, u \rangle \in X$ and $\mathbf{WP}_U(X, Y)$ stands for

$$\mathbf{Prog}_U(X, Y) \wedge \forall Z [\mathbf{Prog}_U(X, Z) \rightarrow Y \subseteq Z]$$

with $\mathbf{Prog}_U(X, Y)$ being $\forall y \in U [\forall z (z \prec_X y \rightarrow z \in Y) \rightarrow y \in Y]$.

By **IARI** + $\bigcup_{i < m} \mathfrak{U}_i$ we denote the theory with only the additional predicates $\mathfrak{U}_0, \dots, \mathfrak{U}_{m-1}$ and their pertaining axioms.

Definition 6.3 Recall the notion of inaccessible set defined in 4.7. For $n > 0$, **Inacc**(n) stands for the set-theoretic statement that there are n -many inaccessible sets $I_0 \in \dots \in I_{n-1}$. β -models were introduced in 4.8. By **Beta**(n) we denote the statement of second order arithmetic asserting that there are n many sets A_0, \dots, A_{n-1} which are β -models of $\Sigma_2^1\text{-AC}$ such that $A_0 \in \dots \in A_{n-1}$, where for sets X, Y of natural numbers $X \in Y$ is defined by $\exists u X = Y_u$.

For $n > 0$, let $\mathbf{MLT}_{nW}V$ be the fragment of **MLTT** with n -many universes $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$, where the W -constructor can solely be applied to families of types in $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ but one can also form the type $V := W_{(A:\mathcal{U}_{n-1})}A$, i.e. a W -type over the largest universe \mathcal{U}_{n-1} . We shall also consider the type theory \mathbf{MLT}_{nW} which is the fragment of $\mathbf{MLT}_{nW}V$ without the type V .

Theorem 6.4 (i) $\mathbf{T}_0^i + \bigcup_{i < n} \mathfrak{U}_i$ has an interpretation in **CZF** + **REA** + **Inacc**(n). The interpretation preserves (at least) all arithmetic statements.

(ii) **CZF** + **REA** + **Inacc**(n) has an interpretation in $\mathbf{MLT}_{nW}V$.

(iii) **CZF** + **REA** + **Inacc**(n) + $\Pi\Sigma W\text{-AC}$ + **RDC** + **PAx** has an interpretation in $\mathbf{MLT}_{nW}^{ext}V$.

(iv) $\mathbf{MLT}_{nW}^{ext}V$ has an interpretation in the classical set theory **KPi** plus an axiom asserting that there exist n -many recursively inaccessible ordinals.

(v) $(\Sigma_2^1\text{-AC}) + \mathbf{BI} + \mathbf{Beta}(n)$ has an interpretation in **KPi** plus the existence of n -many recursively inaccessible ordinals.

(vi) **KPi** plus the existence of n -many recursively inaccessible ordinals has a sets-as-trees interpretation in $(\Sigma_2^1\text{-AC}) + \mathbf{BI} + \mathbf{Beta}(n)$.

(vii) The intuitionistic system **IRA** + $\bigcup_{i < n} \mathfrak{U}_i$ of second order arithmetic can be interpreted in \mathbf{MLT}_{nW} .

(viii) **CZF** + **REA** + **RDC** + **Inacc**(n) has a realizability interpretation in **KPi** plus the existence of n -many recursively inaccessible ordinals.

(ix) All the above theories have the same proof-theoretic strength and prove (at least) the same Π_2^0 -statements of arithmetic.

Proof: The interpretations are extensions of those discussed in the previous section, taking more universes into account. We can only indicate the steps. The interpretation of \mathbf{T}_0^i in $\mathbf{CZF} + \mathbf{REA}$ can be lifted to an interpretation of $\mathbf{T}_0^i + \bigcup_{i < n} \mathbf{U}_i$ into $\mathbf{CZF} + \mathbf{REA} + \mathbf{Inacc}(n)$. The latter theory possesses a sets-as-types interpretation in intensional Martin-Löf type theory with n universes.

$\mathbf{CZF} + \mathbf{REA} + \Pi\Sigma\mathbf{W}\text{-}\mathbf{AC} + \mathbf{RDC} + \mathbf{PAx} + \mathbf{Inacc}(n)$ possesses a sets-as-types interpretation in $\mathbf{MLT}_{nW}^{\text{ext}}\mathbf{V}$. In turn, $\mathbf{MLT}_{nW}^{\text{ext}}\mathbf{V}$ can be interpreted in classical Kripke-Platek set theory \mathbf{KPi} plus an axiom asserting that there are at least n -many recursively inaccessible ordinals following the Ansatz of [59, Theorem 5.11]. $(\Sigma_2^1\text{-}\mathbf{AC}) + \mathbf{BI} + \mathbf{Beta}(n)$ can be easily interpreted in \mathbf{KPi} plus n -recursively inaccessible ordinals.

The proof-theoretic equivalence ensues from an ordinal analysis of the ‘top theory’, \mathbf{KPi} plus the existence of n -many recursively inaccessible ordinals, together with proofs that any ordinal below the proof-theoretic ordinal of that theory is provably well-founded in $\mathbf{T}_0^i + \bigcup_{i < n} \mathbf{U}_i$ as well as $\mathbf{IARI} + \bigcup_{i < n} \mathbf{U}_i$. Neither the ordinal analysis nor the well-ordering proofs are available from the published literature. The ordinal analysis of \mathbf{KPi} plus the existence of n -many recursively inaccessible ordinals, though, can be obtained in a straightforward way by extending the one given for \mathbf{KPi} in [39] or rather its modern version in [13]. It also follows from the ordinal analysis of the much stronger theory \mathbf{KPM} given in [58] by restricting the treatment therein to the pertaining small fragments. For the well-ordering proof substantially more work is required; details will be published in [78]. \square

Theorem 6.5 *The following theories have the same proof-theoretic strength and prove the same Π_2^0 -statements of arithmetic:*

- (i) $\mathbf{T}_0^i + \bigcup_n \mathbf{U}_n$.
- (ii) \mathbf{CZF} plus $\mathbf{Inacc}(n)$ for all $n > 0$.
- (iii) $\mathbf{CZF} + \Pi\Sigma\mathbf{W}\text{-}\mathbf{AC} + \mathbf{RDC} + \mathbf{PAx}$ plus $\mathbf{Inacc}(n)$ for all $n > 0$.
- (iv) The extensional type theory $\mathbf{MLTT}^{\text{ext}}$.
- (v) \mathbf{MLTT} .
- (vi) The classical subsystem of second order arithmetic $(\Sigma_2^1\text{-}\mathbf{AC}) + \mathbf{BI}$ plus $\mathbf{Beta}(n)$ for all $n > 0$.
- (vii) Classical Kripke-Platek set theory \mathbf{KP} plus for every $n > 0$ an axiom asserting that there are at least n -many recursively inaccessible ordinals.
- (viii) $\mathbf{IARI} + \bigcup_n \mathbf{U}_n$.
- (ix) $\mathbf{CZF} + \mathbf{RDC} + \mathbf{LPO}$ plus the axioms $\mathbf{Inacc}(n)$ for all $n > 0$.

Proof: This follows directly from the previous theorem. \square

The latter theorem also shows that the strength of \mathbf{MLTT} is dwarfed by that of $(\Pi_2^1\text{-}\mathbf{CA})$. It corresponds to a tiny fragment of second order arithmetic which itself is a tiny fragment of \mathbf{ZF} , so there are aeons between \mathbf{MLTT} and classical set theory with inaccessible cardinals.

Theorem 6.6 *The following theories prove the same arithmetical statements:*

- (i) $\mathbf{T}_0^i + \bigcup_n \mathbf{U}_n$.
- (ii) \mathbf{MLTT} .
- (iii) The extensional type theory $\mathbf{MLTT}^{\text{ext}}$.
- (iv) \mathbf{CZF} plus $\mathbf{Inacc}(n)$ for every $n > 0$.
- (v) $\mathbf{CZF} + \bigcup_n \mathbf{Inacc}(n) + \bigcup_n \Pi\Sigma\mathbf{W}\text{-}\mathbf{AC} + \mathbf{RDC} + \mathbf{PAx}$.

(vi) $\mathbf{IARI} + \bigcup_n \mathfrak{U}_n$.

Proof: The methods for proving this were described in the proof of 5.9. Details will appear in [78]. \square

Finally, it should be mentioned that Martin-Löf type theory with stronger universes (e.g. Mahlo universes) has been studied by Setzer (cf. [85]).

6.3 Adding the Univalence Axiom

The quote (1) from Harris' book [31] claimed that modeling Voevodsky's univalence axiom (**UA**) requires infinitely many inaccessible cardinals (for a definition of **UA** see [36, Sec. 2.10]). While the simplicial model of type theory with univalence developed in the paper [44] by Kalpulkun, Lumsdaine and Voevodsky is indeed carried out in a background set theory with inaccessible cardinals, it is by no means clear that the existence or proof-theoretic strength of these objects is required for finding a model of type theory with **UA**. In actuality, Bezem, Coquand and Huber in their article [11] provided a cubical model of type theory that also validates **UA**. Crucially, their modeling can be carried out in a constructive background theory such as $\mathbf{CZF} + \bigcup_n \mathbf{Inacc}(n) + \bigcup_n \mathbf{\Pi\Sigma W-AC} + \mathbf{RDC} + \mathbf{PAx}$. Thus it follows that adding **UA** does not increase the strength of type theory and that no inaccessible cardinals are required. Hence in view of Theorem 6.5 we have the following result.

Corollary 6.7 *MLTT has the same proof-theoretic strength as $\mathbf{MLTT} + \mathbf{UA}$. Thus $\mathbf{MLTT} + \mathbf{UA}$ shares the same proof-theoretic strength with all theories listed in Theorem 6.5, in particular with classical Kripke-Platek set theory **KP** augmented by axioms asserting that there are at least n -many recursively inaccessible ordinals for every $n > 0$.*

7 On relating theories III: Omitting W

The proof-theoretic strength of type theories crucially depends on the availability of inductive types and to a much lesser extent on its universes. Relinquishing the W-type brings about an enormous collapse of proof power. Letting \mathbf{MLTT}^- be \mathbf{MLTT} bereft of the W-type constructor, we arrive at a theory no stronger than the system \mathbf{ATR}_0 of reverse mathematics (see [86, I.11]), having the famous ordinal Γ_0 as its proof-theoretic ordinal. According to Feferman's analysis (see [25, 26]), Γ_0 delineates the limit of a notion of predicativity that only accepts the natural numbers as a completed infinity (which was first adumbrated in Hermann Weyl's book "*Das Kontinuum*" from 1918 [94]). Peter Hancock conjectured in the 1970s the ordinal of \mathbf{MLTT}^- to be Γ_0 . Feferman [24] and independently Aczel (see also [1]) proved *Hancock's Conjecture*. There is also a version of \mathbf{CZF} with inaccessible sets of strength Γ_0 , due to Crosilla and Rathjen [18], which does not have set induction. Thus the set-theoretic analogue to eschewing W-types consists in leaving out the principle of set induction. In the next theorem we denote by \mathbf{ATR}_0^i the intuitionistic version of \mathbf{ATR}_0 (see [60, Definition 4.10] for details). By \mathbf{CZF}^- we denote Constructive Zermelo-Fraenkel set theory without set induction but with the Infinity axiom strengthened as follows:

$$0 \in \omega \wedge \forall y [y \in \omega \rightarrow y + 1 \in \omega] \quad (4)$$

$$\forall x [0 \in x \wedge \forall y (y \in x \rightarrow y + 1 \in x) \rightarrow \omega \subseteq x] \quad (5)$$

(for details see [18, Definition 2.2]). Likewise we denote by \mathbf{KP}^- the theory without the set induction scheme but with the infinity axioms (4) and (5).

The notion of weak inaccessibility used below is the one from Definition 4.4. For $n > 0$ let $\mathbf{wInacc}(n)$ be the statement that there exist weakly inaccessible sets x_0, \dots, x_{n-1} such that $x_0 \in \dots \in x_{n-1}$.

A restricted form of **RDC** is Δ_0 -**RDC**: For all Δ_0 -formulae θ and ψ , whenever

$$(\forall x \in a) [\theta(x) \rightarrow (\exists y \in a) (\theta(y) \wedge \psi(x, y))]$$

and $b_0 \in a \wedge \phi(b_0)$, then there exists a function $f : \omega \rightarrow a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega) [\theta(f(n)) \wedge \psi(f(n), f(n+1))].$$

Theorem 7.1 *The following theories share the same proof-theoretic strength and ordinal Γ_0 , and prove the same Π_2^0 -sentences of arithmetic:*

- (i) \mathbf{MLTT}^- .
- (ii) *The extensional version of \mathbf{MLTT}^- .*
- (iii) \mathbf{ATR}_0 .
- (iv) \mathbf{ATR}_0^i .
- (v) $\mathbf{CZF}^- + \forall x \exists y [x \in y \wedge y \text{ is weakly inaccessible} + \Delta_0\text{-RDC}]$.
- (vi) $\mathbf{CZF}^- + \{\mathbf{wInacc}(n) \mid n > 0\} + \mathbf{RDC}$.
- (vii) $\mathbf{KP} + \forall x \exists y [x \in y \wedge y \text{ is admissible}]$.

Proof: We only have to establish that all theories have proof-theoretic ordinal Γ_0 . For extensional \mathbf{MLTT}^- this follows from [24]. The lower bound part, namely that \mathbf{MLTT}^- has at least the strength Γ_0 is due to Jervell [43]. So we are done with (i) and (ii). That \mathbf{ATR}_0 has ordinal Γ_0 is well known. For \mathbf{ATR}_0^i this follows from the observation in [60, Lemma 4.11] that the well ordering proof for any ordinal notation below Γ_0 uses only intuitionistic logic. The determination of the ordinal for the system in (v) and (vi) is due to Rathjen and Crosilla [18, Corollary 9.14] with the validation of $\Delta_0\text{-RDC}$ and \mathbf{RDC} coming from [63, Theorem 4.17] and [63, Theorem 4.16], respectively. The proof-theoretic analysis of the system in (vii) is due to Jäger [38]. \square

We also conjecture that all of the intuitionistic theories from the above list, i.e., \mathbf{MLTT}^- , the extensional version of \mathbf{MLTT}^- , \mathbf{ATR}_0^i , and $\mathbf{CZF}^- + \forall x \exists y [x \in y \wedge y \text{ is weakly inaccessible}]$ prove the same arithmetic statements using the usual techniques. But we have not yet checked that. What is known is that \mathbf{ATR}_0^i embeds in all of these theories (see [60]).

A final question concerns the status of the univalence axiom. Do we get more strength when we add \mathbf{UA} to \mathbf{MLTT}^- ? It turns out that we just have to check whether the cubical model construction from [11] can be carried out in one of the theories from the list. Inspection of [11] reveals that

$$\mathbf{CZF}^- + \forall x \exists y [x \in y \wedge y \text{ is weakly inaccessible}] + \Delta_0\text{-RDC}$$

suffices as a background theory for all the constructions, except W-types.

Corollary 7.2 *The univalent type theory $\mathbf{MLTT}^- + \mathbf{UA}$ is of the same strength as \mathbf{MLTT}^- and \mathbf{ATR}_0 and all the other systems from Theorem 7.1. Therefore its proof-theoretic ordinal is Γ_0 .*

8 Monotone Fixed Point Principles in Intuitionistic Explicit Mathematics

Martin-Löf type theory appears to capture the abstract notion of an inductively defined type very well via its W-type. There are, however, intuitionistic theories of inductive definitions that at first glance appear to be just slight extensions of Feferman's explicit mathematics (see Feferman quote on page 2) but have turned out to be much stronger than anything considered in Martin-Löf type theory. They are obtained from \mathbf{T}_0^i by the augmentation of a monotone fixed point principle which asserts that every monotone operation on classifications (Feferman's notion of set) possesses a least fixed point. To be more precise, there are two versions of this principle. \mathbf{MID} merely postulates the existence of a least solution, whereas \mathbf{UMID} provides a uniform version of this axiom by adjoining a new functional constant to the language, ensuring that a fixed point is uniformly presentable as a function of the monotone operation.

Definition 8.1 For *extensional equality* of classifications we use the shorthand “ $=_{\text{ext}}$ ”, i.e.

$$X =_{\text{ext}} Y = \forall v (v \varepsilon X \leftrightarrow v \varepsilon Y).$$

Further, let $X \subseteq Y$ be a shorthand for $\forall v (v \varepsilon X \rightarrow v \varepsilon Y)$. To state the monotone fixed point principle for subclassifications of a given classification A we introduce the following shorthands:

$$\begin{aligned} \mathbf{Clop}(f, A) & \quad \text{if } \forall X \subseteq A \exists Y \subseteq A fX \simeq Y \\ \mathbf{Ext}(f, A) & \quad \text{if } \forall X \subseteq A \forall Y \subseteq A [X =_{\text{ext}} Y \rightarrow fX =_{\text{ext}} fY] \\ \mathbf{Mon}(f, A) & \quad \text{if } \forall X \subseteq A \forall Y \subseteq A [X \subseteq Y \rightarrow fX \subseteq fY]. \\ \mathbf{Lfp}(Y, f, A) & \quad \text{if } fY \subseteq Y \wedge Y \subseteq A \wedge \forall X \subseteq A [fX \subseteq X \rightarrow Y \subseteq X] \end{aligned}$$

When f satisfies $\mathbf{Clop}(f, A)$, we call f a *classification operation on A* . When f satisfies $\mathbf{Clop}(f, A)$ and $\mathbf{Ext}(f, A)$, we call f *extensional* or an *extensional operation on A* . When f satisfies $\mathbf{Clop}(f, A)$ and $\mathbf{Mon}(f, A)$, we say that f is a *monotone operation on A* . Since monotonicity entails extensionality, a monotone operation is always extensional.

Now we state \mathbf{UMID}_A .

\mathbf{MID}_A (**Monotone Inductive Definition on A**)

$$\forall f [\mathbf{Clop}(f, A) \wedge \mathbf{Mon}(f, A) \rightarrow \exists Y \mathbf{Lfp}(Y, f, A)].$$

\mathbf{UMID}_A (**Uniform Monotone Inductive Definition on A**)

$$\forall f [\mathbf{Clop}(f, A) \wedge \mathbf{Mon}(f, A) \rightarrow \mathbf{Lfp}(\mathbf{lfp}(f), f, A)].$$

\mathbf{UMID}_A states that if f is monotone on subclassifications of A , then $\mathbf{lfp}(f)$ is a least fixed point of f . Let V be the universe, i.e. $V := \{x : x = x\}$. By \mathbf{MID} and \mathbf{UMID} we denote the principles \mathbf{MID}_V and \mathbf{UMID}_V , respectively.⁶

The strength of the various classical versions was determined as a result of several papers [72, 73, 74, 27]. The \mathbf{MID} case is dealt with in [27, 75]. [75] provides a survey of all known results in the classical case. $\mathbf{UMID}_{\mathbb{N}}$ was shown to be related to subsystems of second order arithmetic based on Π_2^1 comprehension. To relate the state of the art in these matters we shall need some terminology. Below we shall distinguish between the classical and the intuitionistic version of a theory by appending the superscript c and i , respectively. For a system S of explicit mathematics we denote by $S \upharpoonright$ the version wherein the induction principles for the natural numbers and for inductive generation are restricted to sets. $\mathbf{IND}_{\mathbb{N}}$ stands for the schema of induction on natural numbers for arbitrary formulas of the language of explicit mathematics. $(\Pi_2^1 - \mathbf{CA})_0$ denotes the subsystem of second order arithmetic (based on classical logic) with Π_2^1 -comprehension but with induction restricted to sets, whereas $(\Pi_2^1 - \mathbf{CA})$ also contains the full schema of induction on \mathbb{N} . [73, 74] yielded the following results:

Theorem 1 (i) $(\Pi_2^1 - \mathbf{CA})_0$ and $\mathbf{T}_0^c \upharpoonright + \mathbf{UMID}_{\mathbb{N}}$ have the same proof-theoretic strength.

(ii) $(\Pi_2^1 - \mathbf{CA})$ and $\mathbf{T}_0^c \upharpoonright + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbb{N}}$ have the same proof-theoretic strength.

The first result about $\mathbf{UMID}_{\mathbb{N}}$ on the basis of intuitionistic explicit mathematics was obtained by Tupailo in [93].

Theorem 2 $(\Pi_2^1 - \mathbf{CA})_0$ and $\mathbf{T}_0^i \upharpoonright + \mathbf{UMID}_{\mathbb{N}}$ have the same proof-theoretic strength.

⁶The acronym for the principle \mathbf{MID} in Feferman’s paper [22], section 7 was $\mathbf{MIG} \upharpoonright$.

[93] uses a characterization of $(\Pi_2^1 - \mathbf{CA})_0$ via a classical μ -calculus (a theory which extends the concept of an inductive definition), dubbed $\mathbf{ACA}_0(\mathcal{L}^\mu)$, given by Möllerfeld [51] and then proceeds to show that $\mathbf{ACA}_0(\mathcal{L}^\mu)$ can be interpreted in its intuitionistic version, $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$, by means of a double negation translation. Finally, as the latter theory is readily interpretable in $\mathbf{T}_0^i \upharpoonright + \mathbf{UMID}_\mathbb{N}$, the proof-theoretic equivalence stated in Theorem 2 follows in view of Theorem 1.

The proof of [93], however, does not generalize to $\mathbf{T}_0^i \upharpoonright + \mathbf{IND}_\mathbb{N} + \mathbf{UMID}_\mathbb{N}$ and extensions by further induction principles. The main reason for this is that adding induction principles such as induction on natural numbers for all formulas to $\mathbf{ACA}_0(\mathcal{L}^\mu)$ only slightly increases the strength of the theory and by no means reaches the strength of $(\Pi_2^1 - \mathbf{CA})$. In order to arrive at a μ -calculus of the strength of $(\Pi_2^1 - \mathbf{CA})$ one would have to allow for transfinite nestings of the μ -operator of length α for any ordinal $\alpha < \varepsilon_0$. As it seems to be already a considerable task to get a clean syntactic formalization of transfinite μ -calculi (let alone furnishing double negation translation thereof), this paper will proceed along a different path. In actuality, much of the work was already accomplished in [73], where it was shown that $(\Pi_2^1 - \mathbf{CA})_0$ and $(\Pi_2^1 - \mathbf{CA})$ can be reduced to operator theories $\mathbf{T}_{<\omega}^{\mathbf{OP}}$ and $\mathbf{T}_{<\varepsilon_0}^{\mathbf{OP}}$, respectively. A careful axiomatization of the foregoing theories in conjunction with results from [72] showed that they lend themselves to double negation translations and thus can be translated into their intuitionistic counterparts. As the intuitionistic theories can be easily viewed as subtheories of $\mathbf{T}_0^i \upharpoonright + \mathbf{UMID}_\mathbb{N}$ and $\mathbf{T}_0^i \upharpoonright + \mathbf{IND}_\mathbb{N} + \mathbf{UMID}_\mathbb{N}$, respectively, one can conclude the following result.

Theorem 3 (i) $(\Pi_2^1 - \mathbf{CA})_0$ and $\mathbf{T}_0^i \upharpoonright + \mathbf{UMID}_\mathbb{N}$ have the same proof-theoretic strength.

(ii) $(\Pi_2^1 - \mathbf{CA})$ and $\mathbf{T}_0^i \upharpoonright + \mathbf{IND}_\mathbb{N} + \mathbf{UMID}_\mathbb{N}$ have the same proof-theoretic strength.

Proof: See [79]. □

Through Theorem 3 one also gets a different proof of Theorem 2 which does not hinge upon [51].

Remark 8.2 Virtually nothing is currently known about the strength of $\mathbf{T}_0^i + \mathbf{MID}$ and variants. In the classical case there is a close relationship with parameter-free Π_2^1 -comprehension. It would be very interesting to investigate whether the strength of \mathbf{MID} diminishes in the intuitionistic setting.

The strength of explicit mathematics with principle like $\mathbf{UMID}_\mathbb{N}$ and even \mathbf{MID} considerably exceeds that of Martin-Löf type theory. This has a bearing on foundational questions such as the limit of constructivity or the limits of different concepts of constructivity. In [76] an attempt is made to delineate the form of constructivism underlying Martin-Löf type theory, suggesting that $\mathbf{T}_0^i + \mathbf{UMID}_\mathbb{N}$ lies beyond its scope.

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